Total Fractional-Order Variation Model

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This review is based on the paper 'A Total Fractional-Order Variation Model for Image Restoration with Non-homogeneous Boundary Conditions and its Numerical Solution' by Jianping Zhang, Ke Chen (2015).

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Outline

Model

Algorithms

A First Look

- **Total $\alpha-$order variation model:**

$$\min_{u \in BV^\alpha(\Omega)} \{E(u) := TV^\alpha(u) + \frac{\lambda}{2} F(u)\}, \ F(u) = \int_\Omega |u - z|^2 dx.$$

Where

$$TV^\alpha(u) := \sup_{\phi \in K} \int_\Omega (-u div^\alpha \phi) dx, \text{ with } div^\alpha \phi = \sum_{i=1}^{d} \frac{\partial \phi_i}{\partial x_i^\alpha}$$

- When $u \in W_1^\alpha(\Omega)$, $TV^\alpha(u) = \int_\Omega |\nabla^\alpha u| dx$.

Fractional Derivative

Theorem

The Abels integral equation, with,

$$\frac{1}{\Gamma(\alpha)}\int_0^x \frac{\psi(\tau)}{(x-\tau)^{1-\alpha}}d\tau = f(x),\ x>0$$

has the solution given by the formula

$$\psi(x) = \frac{1}{\tau(1-\alpha)}\frac{d}{dx}\int_0^x \frac{f(\tau)}{(x-\tau)}d\tau,\ x>0.$$

With the help of this theorem, we can then define the left, right sided and the central Riemann-Liouville (R-L) derivative:

$$D^{\alpha}_{[a,x]}f(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha-n+1}}$$

$$D^{\alpha}_{[x,b]}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_a^x \frac{f(\tau)}{(\tau-x)^{\alpha-n+1}}$$

$$D^{\alpha}_{[a,b]}f(x) = \frac{1}{2}(D^{\alpha}_{[a,x]}f(x) + (-1)^n D^{\alpha}_{[x,b]}f(x))$$

Other Fractional Derivative Definitions

- Grunwald-Letnikov (G-L) left-sided derivative is denoted by:

$$GD^{\alpha}_{[a,x]}f(x) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{[\frac{x-a}{h}]}(-1)^j\binom{\alpha}{j}f(x-jh),$$

where $\binom{\alpha}{j} = \dfrac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!},$

- The third definition is the Caputo order α derivative defined by

$$CD^{\alpha}_{[a,x]}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}}$$

$$CD^{\alpha}_{[x,b]}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(\tau)}{(\tau-x)^{\alpha-n+1}}$$

$$CD^{\alpha}_{[a,b]}f(x) = \frac{1}{2}(CD^{\alpha}_{[a,x]}f(x) + (-1)^n CD^{\alpha}_{[x,b]}f(x))$$

Properties

- Linearity.

$$D^{\alpha}_{[a,x]}(pf(x)+qg(x)) = pD^{\alpha}_{[a,x]}f(x) + qD^{\alpha}_{[a,x]}g(x)$$

- Singularity. Assume that$D^{\alpha}_{[a,x]}$ is one of the above three fractional-order derivative operators. For any non-integer $\alpha > 0$ and $x > a$, there exists a non-constant value function $f(\tau)$ in $(a, x]$ such that $D^{\alpha}_{[a,x]}f(x)$. For example, $f(\tau) = (x - 2\tau)(x - \tau)^{\alpha}$ will give $D^{\alpha}_{[0,x]}f(x) = 0$ in Abel's inverse transform.
- Boundary conditions: we need to assume **zero** Dirichlet boundary condition, otherwise there is a singularity at the end point. However, if we have nonzero Dirichlet boundary conditions, one solution could be to extract off a linear approximation $g(x)$ (that coincides with f at $x = a, b$) and to consider $D^{\alpha}_{[a,x]}(f(x) - g(x))$.

Theoretical Results

We may skip the third part of the paper (3 pages), which clarifies some theoretical/classical results. Basically, it says W_p^α embedded with $\|\cdot\|_{W_p^\alpha}$ is a Banach space (this is basic guarantee of any iterative approximation algorithms) and the proposed model has a unique minimizer (convexity of energy functional $E(u)$).

Boundary Conditions

In order to use deal with non-zero Dirichlet boundary conditions (which is almost always the case in real world), an auxiliary function is introduced.

- In 1D case, if $u(0) = a$, $u(1) = b$, then introduce $e(x) = a(1-x) + bx$ and take $\hat{u}(x) = u(x) - e(x)$, then we achieve

$$\hat{u}(0) = \hat{u}(1) = \hat{u}'(0) = \hat{u}'(1) = 0.$$

- In 2D, assuming that the four corners of the solution are given or accurately estimated:

$$u(0,0) = a, u(0,1) = b, u(1,0) = c, u(1,1) = d.$$

  Similarly we can construct a bilinear function, $e_1(x,y) = a + (c-a)x + (b-a)y + (d+a-c-b)xy$, then $\hat{u}(x,y) = u(x,y) - e_1(x,y)$ will have zero values on all 4 corners. If boundary conditions $u(0,y) = a_1(y)$, $u(1,y) = a_2(y)$, $u(x,0) = b_1(x)$, $u(x,1) = b_2(x)$ at $\partial\Omega$ are known as a priori, then we define $e_2(x,y) = ((1-x)\hat{a}(y) + x\hat{a}(y)) + ((1-y)\hat{b}(x) + y\hat{b}(x))$, then we achieve

$$\hat{u}(x,y)|_{\partial\Omega} = 0, \text{ where } \hat{u}(x,y) = u(x,y) - e_1(x,y) - e_2(x,y).$$

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Discretization

Key Point: Discrete version of fractional derivative. It turns out the x-direction

$$D^{\alpha}_{[a,b]}f(x_k,y_l)=\frac{\delta_0^{\alpha}f(x_k,y_l)}{h^{\alpha}}+O(h)=\frac{1}{2}(\frac{\delta_-^{\alpha}f(x_k,y_l)}{h^{\alpha}}+\frac{\delta_+^{\alpha}f(x_k,y_l)}{h^{\alpha}})+O(h)$$
$$=\frac{1}{2}(h^{-\alpha}\sum_{j=0}^{k+1}w_j^{\alpha}f^l_{k-j+1}+h^{-\alpha}\sum_{j=0}^{N-k+2}w_j^{\alpha}f^l_{k+j-1})+O(h)$$

is applicable to both R-L and C derivatives, where $f_s^l=f(x_s,y_l)$, $w_j^{(\alpha)}=(-1)^j\binom{\alpha}{j}$, j = 0, 1,..., N+1.

Matrix Version

Denote $w = w_0^\alpha + w_2^\alpha$, then we can write $D_{[a,b]}^\alpha f(x_k, y_l)$ into matrix:

$$\begin{pmatrix} \delta_0^\alpha f(x_1, y_l) \\ \delta_0^\alpha f(x_2, y_l) \\ \vdots \\ \vdots \\ \delta_0^\alpha f(x_N, y_l) \end{pmatrix} = \underbrace{\frac{1}{2h^\alpha} \begin{pmatrix} 2\omega_1^\alpha & w & \omega_3^\alpha & \cdots & \omega_N^\alpha \\ w & 2\omega_1^\alpha & \ddots & \ddots & \vdots \\ \omega_3^\alpha & \ddots & \ddots & \ddots & \omega_3^\alpha \\ \vdots & \ddots & \ddots & 2\omega_1^\alpha & w \\ \omega_N^\alpha & \cdots & \omega_3^\alpha & w & 2\omega_1^\alpha \end{pmatrix}}_{B_N^\alpha} \underbrace{\begin{pmatrix} f_1^l \\ f_2^l \\ \vdots \\ \vdots \\ f_N^l \end{pmatrix}}_{f}.$$

Review the model:

$$\min_{u,d} \{ \int_\Omega |d| dx + \frac{\lambda}{2} F(u) \}, \text{ s.t. } F(u) = \int_\Omega |u - z|^2 dx, d = \nabla^\alpha(u).$$

and discrete matrix version of TV:

$$\min_u \frac{\lambda}{2} \|u - z\|^2 + \|D_1 u\| + \|D_2 u\|$$

Now, it's clear, whenever you see D just replace it with B!

Discussions?